Long periodic waves on an even beach

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High-order Boussinesq-type equations for long waves over an even slope are derived and investigated. Potential and surface elevation for periodic wave motion are expanded in Fourier series up to the third harmonic inclusively. Coefficients of this series are expressed as polynomials of Bessel functions.

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I. INTRODUCTION

A special feature of Boussinesq-type equations is a possibility to reduce the dimension of a problem by expanding the velocity potential as a power series in the vertical coordinate. This expansion had been used by Lagrange [1], developed by Boussinesq [2], and received its modern form in the paper of Friedrichs [3].

In 1966, Mei and Méhauté [4] extended these equations to an uneven bottom in one dimension using the bottom potential as a basic variable. Later, similar equations based on the depth-averaged velocity and on the velocity at the still water level were derived (see survey in the paper of Madsen and Schäffer [5]). In the present paper, we follow Mei and Méhauté and write down Boussinesq-type equations for the bottom potential.

There are two small parameters associated with Boussinesq-type equations: the ratio of amplitude to depth ε and the ratio of depth to wavelength μ. The classical Boussinesq equations include terms of order $O(\varepsilon)$, $O(\mu^2)$ and assume that $O(\varepsilon)=O(\mu^2)$. We work in the next order using the same assumption, so our equations include ε^2, $\varepsilon\mu^2$, and μ^4 terms.

We consider a regular periodic wave motion over an even slope sx excluding the deep-water region where the shallow-water restrictions are violated and the neighborhood of shoreline where singularity is possible. The potential at the bottom is expanded in Fourier series $\Sigma_{m=1}^{3}S^m(x)\sin(m\omega t)$ up to orders $(\varepsilon^2,\ \varepsilon\mu^2,\mu^4)$ where functions $S^m(x)$ are homogenous polynomials of Bessel functions $Z_0(2\omega\sqrt{x/s})$ and $Z_1(2\omega\sqrt{x/s})$ whose coefficients are Loran polynomials of $\sqrt{x}$. (The first term of this expansion $J_0(2\omega\sqrt{x/s})\sin(\omega t)$ is used, for example, in the work of Mei [6]). Expansion for the surface elevation is also given.

We conjecture that Boussinesq-type equations can be written and solutions of the specified form can be calculated for an arbitrary set of orders of kept terms.

II. BASIC EQUATIONS

Nondimensional coordinates are used as follows:

$$x=\frac{x'}{l'_0},\quad z=\frac{z'}{h'_0},\quad t=\frac{g^{1/2}h_0'^{\,1/2}}{l'_0}t',\quad \eta=\frac{\eta'}{a'_0},$$
$$\varphi=\frac{h'_0}{a'_0 l'_0 g^{1/2}h_0'^{\,1/2}}\varphi',\quad h=\frac{h'}{h'_0},\tag{1}$$

where the prime denotes physical variables and a'_0, l'_0, and h'_0 denote characteristic wave amplitude, depth, and wavelength, respectively. The scaled governing equation and boundary conditions for the irrotational wave problem read

$$\mu^2\varphi_{xx}+\varphi_{zz}=0,\quad -h(x)<z<\varepsilon\eta(x,t),\tag{2}$$

$$\eta_t+\varepsilon\varphi_x\eta_x-\mu^{-2}\varphi_z=0,\quad z=\varepsilon\eta(x,t),\quad (A),\tag{3}$$

$$\varphi_t+\frac{1}{2}\varepsilon(\varphi_x^2+\mu^{-2}\varphi_z^2)+\eta=0,\quad z=\varepsilon\eta(x,t),\quad (B),\tag{4}$$

$$\varphi_z=-\mu^2h_x\varphi_x,\quad z=-h(x),\tag{5}$$

where ε and μ are the measures of nonlinearity and frequency dispersion defined by

$$\varepsilon=a'_0/h'_0,\quad \mu=h'_0/l'_0.\tag{6}$$

We expand the potential $\varphi(x,z,t)$ in powers of a vertical coordinate

$$\varphi(x,z,t)=\sum_{m=0}^{\infty}[z+h(x)]^mF_m(x,t)\tag{7}$$

and assume that the function defining the bottom $z=-h(x)$ is of linear form

$$h(x)\equiv sx.\tag{8}$$

[Owing to (8), the linear part of Eq. (15) is of Bessel type.]

Substituting (7) and (8) into (2) and equating to zero coefficients of each power of $z+h(x)$, we have

$$F_{m+2}=-\mu^2\frac{2s(m+1)F_{m+1,x}+F_{m,xx}}{(m+2)(m+1)(1+s^2\mu^2)}.\tag{9}$$

The boundary condition at the bottom (5) gives

$$F_1=-\mu^2\frac{2sF_{0,x}}{1+s^2\mu^2}.\tag{10}$$

Denoting $f(x,t)\equiv F_0(x,t)$, and expanding all expressions in powers of μ, we obtain the first terms of φ

$$\varphi=f+\mu^2(z+h)f_x(-s+s^3\mu^2-s^5\mu^4)+\mu^2(z+h)^2f_{xx}\times\left(-\frac{1}{2}+\frac{3}{2}s^2\mu^2-\frac{5}{2}s^4\mu^4\right)+\mu^4(z+h)^3f_{xxx}\times\left(\frac{1}{2}s-\frac{5}{3}s^3\mu^2\right)+\mu^4(z+h)^4f_{xxxx}\left(\frac{1}{24}-\frac{5}{12}s^2\mu^2\right)+\mu^6(z+h)^5f_{xxxxx}\left(-\frac{1}{24}s\right)+\mu^6(z+h)^6f_{xxxxxx}\times\left(-\frac{1}{720}\right). \tag{11}$$

Expression (11) satisfies (2) and (5). Substitution of Eq. (11) into (3) and (4) gives the Boussinesq-type equations for potential at bottom $f(x,t)$ and surface elevation $\eta(x,t)$:

$$\eta_t+sf_x+sxf_{xx}+\left[-s^3f_x-3s^3xf_{xx}-\frac{3}{2}s^3x^2f_{xxx}-\frac{1}{6}s^3x^3f_{xxxx}\right]\mu^2+\left\{f_x+\left[-s^2f_x-2s^2xf_{xx}-\frac{1}{2}s^2x^2f_{xxx}\right]\mu^2\right\}\eta_x\varepsilon+\left\{f_{xx}+\left[-3s^2f_{xx}-3s^2xf_{xxx}-\frac{1}{2}s^2x^2f_{xxxx}\right]\mu^2\right\}\eta\varepsilon+\left[s^5f_x+5s^5xf_{xx}+5s^5x^2f_{xxx}+\frac{5}{3}s^5x^3f_{xxxx}+\frac{5}{24}s^4x^4f_{xxxxx}+\frac{1}{120}s^5x^5f_{xxxxxx}\right]\mu^4=0, \tag{12}$$

$$\eta+f_t+\left[-s^2xf_{xt}-\frac{1}{2}s^2x^2f_{xxt}\right]\mu^2+\left[\frac{1}{2}f_x^2\right]\varepsilon+\left[-\frac{1}{2}s^2f_x^2-s\eta f_{xt}-s^2xf_xf_{xx}+\frac{1}{2}s^2x^2f_{xx}^2-sx\eta f_{xxt}-\frac{1}{2}s^2x^2f_xf_{xxx}\right]\varepsilon\mu^2+\left[s^4xf_{xt}+\frac{3}{2}s^4x^2f_{xxt}+\frac{1}{2}s^4x^3f_{xxxt}+\frac{1}{24}s^4x^4f_{xxxxt}\right]\mu^4=0. \tag{13}$$

To express the surface elevation $\eta(x,t)$ in terms of f and its derivatives, we expand it in powers of μ: $\eta=\eta_0+\eta_2\mu^2+\eta_4\mu^4+O(\mu^6)$. (Expansion of η_i in powers of ε is not important for this purpose.) After substitution of this expansion in (13), the following formulas are derived:

$$\eta_0=-f_t-\frac{1}{2}f_x^2\varepsilon,$$

$$\eta_2=s^2xf_{xt}+\frac{1}{2}s^2x^2f_{xxt}+\left[\frac{1}{2}s^2f_x^2-sf_tf_{xt}+s^2xf_xf_{xx}-\frac{1}{2}s^2x^2f_{xx}^2-sxf_tf_{xxt}+\frac{1}{2}s^2x^2f_xf_{xxx}\right]\varepsilon,$$

$$\eta_4=-s^4xf_{xt}-\frac{3}{2}s^4x^2f_{xxt}-\frac{1}{2}s^4x^3f_{xxxt}-\frac{1}{24}s^4x^4f_{xxxxt}. \tag{14}$$

Substituting (14) in (12), we have the single equation for the function f:

$$-f_{tt}+sf_x+sxf_{xx}+\left[-s^3f_x+s^2xf_{xtt}-3s^3xf_{xx}+\frac{1}{2}s^2x^2f_{xxtt}-\frac{3}{2}s^3x^2f_{xxx}-\frac{1}{6}s^3x^3f_{xxxx}\right]\mu^2+[-f_{xx}f_t-2f_xf_{xt}]\varepsilon+\left[s^5f_x-s^4xf_{xtt}+5s^5xf_{xx}-\frac{3}{2}s^4x^2f_{xxtt}+5s^5x^2f_{xxx}-\frac{1}{2}s^4x^3f_{xxxtt}+\frac{5}{3}s^5x^3f_{xxxx}-\frac{1}{24}s^4x^4f_{xxxxtt}+\frac{5}{24}s^5x^4f_{xxxxx}+\frac{1}{120}s^5x^5f_{xxxxxx}\right]\mu^4+\left[-sf_{tt}f_{xt}+3s^2f_tf_{xt}-sf_tf_{xtt}+3s^2f_tf_{xx}+4s^2xf_{xx}f_{xt}-sxf_{tt}f_{xxt}+3s^2xf_xf_{xxt}-\frac{1}{2}s^2x^2f_{xx}f_{xxt}-sxf_tf_{xxtt}+3s^2xf_tf_{xxx}+s^2x^2f_{xt}f_{xxx}+s^2x^2f_xf_{xxxt}+\frac{1}{2}s^2x^2f_{xxxx}f_t\right]\varepsilon\mu^2+\left[\frac{3}{2}f_x^2f_{xx}\right]\varepsilon^2=0. \tag{15}$$

III. PERIODIC PROBLEM

We suppose that the solution is periodic in time and can be expanded in a Fourier series in an area excluding the deep-water region and the neighborhood of shoreline

$$f(x,t)=[S_{00}^1(x)+S_{20}^1(x)\varepsilon^2+S_{02}^1(x)\mu^2+S_{04}^1(x)\mu^4]\sin(\omega t)+[S_{10}^2(x)\varepsilon+S_{12}^2(x)\varepsilon\mu^2]\sin(2\omega t)+S_{20}^3(x)\varepsilon^2\sin(3\omega t). \tag{16}$$

[Forms of coefficients near $\sin(m\omega t)$ are determined by recurrent calculations when solving (15)].

Denote by $Z=Z(x)$ a solution to the equation

$$\omega^2Z+sZ_x+sxZ_{xx}=0 \tag{17}$$

and by Z' its derivative. $Z(x)$ and $Z'(x)$ can be expressed in terms of Bessel functions in the following way:

$$Z(x)=\alpha J_0\left(2\omega\sqrt{\frac{x}{s}}\right)+\beta Y_0\left(2\omega\sqrt{\frac{x}{s}}\right),\quad Z'(x)=\omega s^{-1/2}x^{-1/2}\left[-\alpha J_1\left(2\omega\sqrt{\frac{x}{s}}\right)-\beta Y_1\left(2\omega\sqrt{\frac{x}{s}}\right)\right]. \tag{18}$$

The major finding of this paper is the following expressions for $S^i_{\alpha\beta}$:

$$S^1_{00}=Z, \tag{19}$$

$$S^1_{20}=-\frac{\omega^4}{8s^3x}Z^3-\frac{\omega^2}{8s^2x}Z^2Z'-\frac{\omega^2}{8s^2}ZZ'^2-\frac{1}{8s}Z'^3, \tag{20}$$

$$S^1_{02}=-\frac{2s\omega^2x}{9}Z+\left(\frac{7s^2x}{9}+\frac{s\omega^2x^2}{9}\right)Z', \tag{21}$$

$$S^1_{04}=\left(-\frac{479s^3\omega^2x}{1350}-\frac{29s^2\omega^4x^2}{225}-\frac{s\omega^6x^3}{162}\right)Z+\left(-\frac{479s^4x}{1350}-\frac{136s^3\omega^2x^2}{675}-\frac{7s^2\omega^4x^3}{450}\right)Z', \tag{22}$$

$$S^2_{10}=-\frac{\omega}{2s}ZZ', \tag{23}$$

$$S^2_{12}=\left(\frac{4\omega^3}{3}+\frac{7\omega^5x}{18s}\right)Z^2+\frac{7\omega^3x}{3}ZZ'+\left(-\frac{2s\omega x}{9}-\frac{7\omega^3x^2}{18}\right)Z'^2, \tag{24}$$

$$S^3_{20}=-\frac{\omega^4}{8s^3x}Z^3-\frac{\omega^2}{8s^2x}Z^2Z'+\frac{3\omega^2}{8s^2}ZZ'^2+\frac{1}{24s}Z'^3. \tag{25}$$

Substituting these expressions in (14), we obtain the following expressions for $\eta(x,t)$:

$$\eta(x,t)=C^0_{10}(x)\varepsilon+C^0_{12}(x)\varepsilon\mu^2+[C^1_{00}(x)+C^1_{20}(x)\varepsilon^2+C^1_{02}(x)\mu^2+C^1_{04}(x)\mu^4]\cos(\omega t)+(C^2_{10}(x)\varepsilon+C^2_{12}(x)\varepsilon\mu^2)\cos(2\omega t)+C^3_{20}(x)\varepsilon^2\cos(3\omega t), \tag{26}$$

where

$$C^0_{10}=-\frac{1}{4}Z'^2, \tag{27}$$

$$C^0_{12}=\frac{\omega^4}{4}Z^2+\frac{\omega^4x}{18}ZZ'-\frac{7s\omega^2x}{36}Z'^2, \tag{28}$$

$$C^1_{00}=-\omega Z, \tag{29}$$

$$C^1_{20}=\frac{\omega^5}{8s^3x}Z^3-\frac{\omega^3}{8s^2x}Z^2Z'+\left(\frac{\omega^3}{8s^2}-\frac{\omega}{4sx}\right)ZZ'^2+\frac{3\omega}{8s}Z'^3, \tag{30}$$

$$C^1_{02}=-\frac{5s\omega^3x}{18}Z+\left(-\frac{5s^2\omega x}{18}-\frac{s\omega^3x^2}{9}\right)Z', \tag{31}$$

$$C^1_{04}=\left(\frac{283s^3\omega^3x}{2700}-\frac{43s^2\omega^5x^2}{1800}+\frac{s\omega^7x^3}{162}\right)Z+\left(\frac{283s^4\omega x}{2700}-\frac{103s^3\omega^3x^2}{1350}-\frac{s^2\omega^5x^3}{25}\right)Z', \tag{32}$$

$$C^2_{10}=\frac{\omega^2}{s}ZZ'+\frac{1}{4}Z'^2, \tag{33}$$

$$C^2_{12}=\left(-\frac{23\omega^4}{12}-\frac{7\omega^6x}{9s}\right)Z^2-\frac{49\omega^4x}{18}ZZ'+\left(\frac{41s\omega^2x}{36}+\frac{7\omega^4x^2}{9}\right)Z'^2, \tag{34}$$

$$C^3_{20}=\frac{3\omega^5}{8s^3x}Z^3+\frac{5\omega^3}{8s^2x}Z^2Z'+\left(\frac{\omega}{4sx}-\frac{9\omega^3}{8s^2}\right)ZZ'^2-\frac{3\omega}{8s}Z'^3. \tag{35}$$

IV. CONCLUSIONS

A solution with the precision of (ε^2, $\varepsilon\mu^2$, μ^4) to Eqs. (2)–(5) is presented. The intermediate equations are given for illustrating the method of derivation but the expressions (19)–(33) can be checked by substitution into the system (2)–(5) [using expression (11) for the potential].

We conjecture that these expressions are only the first terms of some expanded exact solution to system (2)–(5).

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